STABILITY CONDITIONS FOR THE STEADY-STATE ROTATION OF A CYLINDER FILLED WITH A STRATIFIED NONUNIFORM VISCOUS INCOMPRESSIBLE LIQUID

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The problem of the stability of the steady-state rotation of a solid with a cavity partly filled with a uniform viscous incompressible liquid occurs in the analysis of models of various kinds of turbine machinery (see, for example, [1-3]). In [4] there was an experimental investigation of the motion of a gyroscope with a cavity containing a stratified nonuniform viscous liquid.

In the present study we use the method of [3] to solve the plane problem of the stability in the small of the steady-state rotation of a circular cylinder completely filled with two immiscible viscous incompressible liquids, in the case of axially symmetric viscoelastic gripping of the cylinder axis when it rotates when constant angular velocity.

1. Statement of the Problem. Suppose that a circular cylinder with inner radius a, completely filled with two immiscible viscous incompressible liquids with densities ρ_1, ρ_2 and viscosities μ_1, μ_2, is rotating in the steady state with angular velocity Ω about the axis O_1z. In the steady-state rotation condition the axis of the cylinder, which is viscoelastically gripped, coincides with O_1z, and the liquids filling the cylinder rotate with it like a solid, where the interface between them is a cylindrical surface of radius b with axis O_1z and the less dense liquid (with density ρ_2) is in the central part of the cylinder.

We shall consider the problem of the stability of the steady-state rotation of the cylinder with liquid filling in a linear approximation and in the framework of a plane model, i.e., assuming that in the perturbed motion the cylinder and the liquids filling it undergo plane-parallel displacements perpendicular to the axis O_1z, where the field of velocities, the pressure, and the density are independent of the coordinate along this axis. The absolute angular velocity of the rotation of the cylinder in the perturbed motion will be considered constant and equal to Ω.

We introduce the fixed rectangular coordinate system $O_1x_1x_2z$ (O_1z is the axis of steady-state rotation of the cylinder). The system of linearized equations of the plane model and the boundary conditions associated with them include: 1) the equations of the translational motion of the cylinder parallel to the plane $O_1x_1x_2$

$$M\ddot{x}_j^0 + H\dot{x}_j^0 + Kx_j^0 = F_j, \quad j = 1, 2, \tag{1.1}$$

where the x_j^0 are the coordinates of the point of intersection of the cylinder axis with the plane $O_1x_1x_2$; the F_j are the components of the force with which the liquid filling acts upon a unit length of the cylinder; M is the mass of a unit length of the cylinder; H and K are, respectively, the coefficients of damping and rigidity of gripping of the cylinder axis divided by its length; 2) the condition Ω = const; 3) the equations of motion of the viscous incompressible liquid in the plane $O_1x_1x_2$, linearized near the steady-state quasisolid rotation of the liquid about the axis O_1z; 4) the condition for the adhesion of the liquid to the inner surface of the cylinder, carried over in a linear approximation on the basis of the deviations from the state of steady-state rotation onto the surface $x_1^2 + x_2^2 = a^2$; 5) continuity of the velocities and the stresses and the kinematic condition on the interface between the liquids in the linear approximation on the basis of the deviations from the state of steady-state quasisolid rotation of the liquids; 6) expressions determining the components of the force with which the liquid filling acts on a unit length of the cylinder.

The above-listed equations in the deviations from steady-state rotation and the boundary conditions associated with them admit of particular solutions proportional to $e^{\lambda t}$, where λ is an eigenvalue. The steady-state rotation of the cylinder with the liquid filling will be considered stable in the small if all the λ have negative real parts and will be considered

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unstable if at least one λ has a positive real part.

2. Method of Investigating the Stability. If the eigenvalues are continuous functions of the parameters of the problem, a change in the degree of instability in the system takes place when an imaginary λ occurs. Since the system 1-6 is invariant under the coordinate transformation $x_1' = x_2$, $x_2' = -x_1$, we can show as in [3] that all the values of the parameters for which there exists an imaginary eigenvalue $\lambda = i\omega$ can be found from the conditions of realizability of circular precession of a cylinder with liquid filling, i.e., such a motion that the point of intersection of the cylinder axis with the plane $O_1x_1x_2$ describes a circle with frequency ω and the deviations of the hydrodynamic elements from the steady-state values vary with time proportionally to $e^{i\omega t}$.

Suppose that the rotating cylinder filled with two immiscible viscous incompressible liquids undergoes circular precession with frequency ω. We introduce the noninertial measurement system $O\xi\eta$, bound to the line of centers O_1O (Fig. 1). As is shown in the appendix, in the case of circular precession, the field of velocities, the pressure, and the position of the interface between the liquids in the system $O\xi\eta$ will be independent of time.

Thus, solving the problem of steady-state motion of the liquids in the system $O\xi\eta$, calculating the force with which the liquid filling acts on the cylinder in the case of circular precession, we can use the equations of translational motion of the cylinder, (1.1), to find the conditions imposed on the problem parameters for which it is possible to have circular precession of the cylinder with the liquid filling. These conditions determine the boundaries of the regions with different degrees of instability in the space of problem parameters, in particular the boundary of the region of stability.

3. The Hydrodynamic Problem. Suppose that an infinite cylinder with inner radius a, completely filled with two immiscible liquids, is rotating with absolute angular velocity Ω about its axis and undergoing precession along a circle of small radius ε with frequency ω (see Fig. 1). On the inner surface of the cylinder there is a layer of uniform incompressible liquid with density ρ_1 and viscosity μ_1, and in its central part there is a liquid with density ρ_2 and viscosity μ_2.

When $\varepsilon = 0$, we have a steady-state motion of the liquids in which they rotate like a solid with angular velocity Ω about the axis of the cylinder. In this case the interface between the liquids is a cylindrical surface of radius b which is coaxial with the rotating cylinder.

In the noninertial system $O\xi\eta$, which is rigidly bound to the line of centers O_1O, we introduce the coordinates r and φ (see Fig. 1). We consider the steady-state motions of the liquid in the plane $O\xi\eta$ which are close to quasisolid rotation of the kind which the liquid can undergo when $\varepsilon = 0$:

$$u' = 0,\ v' = (\Omega - \omega)r, \tag{3.1}$$

where u' and v' are the radial and azimuthal components of the velocity in the system $O\xi\eta$.

The Navier–Stokes equations for steady-state motions in the plane $O\xi\eta$, linearized near (3.1), can be written in the form

$$\begin{gathered}
(\Omega-\omega)\frac{\partial u_k}{\partial\varphi} = \Omega^2 r + \omega^2\varepsilon\cos\varphi + 2\Omega v_k - \frac{1}{\rho_k}\frac{\partial p_k}{\partial r} + \nu_k\left(\Delta u_k - \frac{2}{r^2}\frac{\partial v_k}{\partial\varphi} - \frac{u_k}{r^2}\right),\\
(\Omega-\omega)\frac{\partial v_k}{\partial\varphi} = -2\Omega u_k - \omega^2\varepsilon\sin\varphi - \frac{1}{\rho_k r}\frac{\partial p_k}{\partial\varphi} + \nu_k\left(\Delta v_k + \frac{2}{r^2}\frac{\partial u_k}{\partial\varphi} - \frac{v_k}{r^2}\right),\\
\frac{\partial u_k}{\partial r} + \frac{u_k}{r} + \frac{1}{r}\frac{\partial v_k}{\partial\varphi} = 0,
\end{gathered} \tag{3.2}$$

where u_k and v_k are the deviations of the radial and azimuthal components of the velocity field of the liquid from the corresponding components of the field (3.1); p_k is the pressure; ν_k is the kinematic viscosity of the liquid; k = 1, 2.

The boundary condition of adhesion of the liquid to the surface of the cylinder is of the form

$$u_1 = v_1 = 0,\ r = a. \tag{3.3}$$

The continuity conditions for the velocities and stresses and the kinematic condition on the interface between the liquids $r = b + \eta(\varphi)$ in a linear approximation with respect to

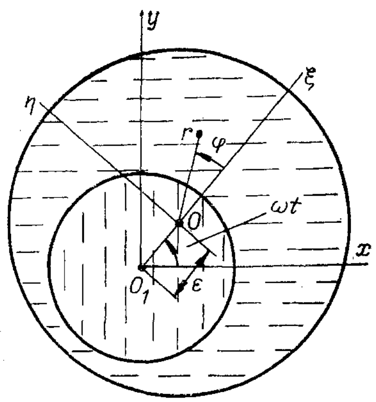

Fig. 1

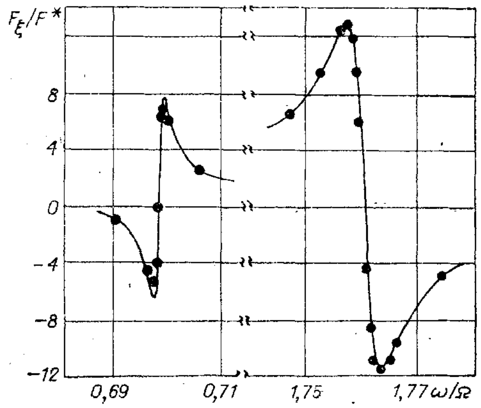

Fig. 2

ε can be written as

$$p_2 - p_1 + \Omega^2 b\eta(\rho_2 - \rho_1) + 2\left(\mu_1 \frac{\partial u_1}{\partial r} - \mu_2 \frac{\partial u_2}{\partial r}\right) = 0,$$

$$\mu_1\left(\frac{1}{r}\frac{\partial u_1}{\partial \varphi} + \frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right) = \mu_2\left(\frac{1}{r}\frac{\partial u_2}{\partial \varphi} + \frac{\partial v_2}{\partial r} - \frac{v_2}{r}\right), \tag{3.4}$$

$$u_1 = u_2, \quad v_1 = v_2, \quad (\Omega - \omega)\frac{\partial \eta}{\partial \varphi} = u_1, \quad r = b.$$

To (3.3) and (3.4) we should also adjoin the condition of boundedness of the solution of the system (3.2) when r = 0.

4. The Exact Solution. The boundary-value problem (3.2)-(3.4) admits of an exact solution. Introducing the Lamb potentials θ_k, ψ_k by the formulas

$$u_k = \frac{\partial \theta_k}{\partial r} + \frac{1}{r}\frac{\partial \psi_k}{\partial \varphi}, \quad v_k = \frac{1}{r}\frac{\partial \theta_k}{\partial \varphi} - \frac{\partial \psi_k}{\partial r}, \quad k = 1, 2, \tag{4.1}$$

we find, making use of the same methods as in [3]:

$$\theta_1 = 2\,\mathrm{Re}\left[\left(c_{11}r + \frac{c_2}{r}\right)e^{i\varphi}\right], \quad \theta_2 = 2\,\mathrm{Re}\,(c_{21}re^{i\varphi}),$$

$$\psi_1 = 2\,\mathrm{Re}\left\{\left[-\frac{2\Omega}{\Omega-\omega}i\left(c_{11}r + \frac{c_2}{r}\right) + c_{13}L_1(\lambda_1 r) + c_4 M_1(\lambda_1 r)\right]e^{i\varphi}\right\},$$

$$\psi_2 = 2\,\mathrm{Re}\left\{\left[-\frac{2\Omega}{\Omega-\omega}ic_{21}r + c_{23}R_1(\lambda_2 r)\right]e^{i\varphi}\right\}, \tag{4.2}$$

$$p_k = \rho_k\left[-(\Omega-\omega)\frac{\partial \theta_k}{\partial \varphi} + \frac{\Omega^2}{2}(r^2 - b^2) + \omega^2 \varepsilon r\cos\varphi - 2\Omega\psi_k\right],$$

$$\eta(\varphi) = 2\,\mathrm{Re}\,(\eta_* e^{i\varphi}),$$

where $L_n(\lambda_1 r) = e^{-\varkappa_1 a}H_n^{(2)}(\lambda_1 r)$; $M_n(\lambda_1 r) = e^{\varkappa_1 b}H_n^{(1)}(\lambda_1 r)$; $R_n(\lambda_2 r) = 2e^{-\varkappa_2 b}J_n(\lambda_2 r)$; $\lambda_k = \varkappa_k\left(-\frac{\Omega-\omega}{|\Omega-\omega|} + i\right)$; $\varkappa_k = (|\Omega-\omega|/2\nu_k)^{1/2}$; $H_n^{(1),(2)}$, J_n are, respectively, the Hankel and Bessel functions. The constants c_{11}, c_{21}, c_2, c_{13}, c_4, η_*, c_{23} in the solution (4.2) satisfy a system of linear algebraic equations:

$$\frac{3-\tau}{1-\tau}c_{11} + \frac{1+\tau}{1-\tau}\frac{c_2}{a^2} + \frac{i}{a}Z_1(\lambda_1 a) = 0,$$

$$\frac{3-\tau}{1-\tau}ic_{11} + \frac{1+\tau}{1-\tau}i\frac{c_2}{a^2} - \lambda_1 Z_0(\lambda_1 a) + \frac{1}{a}Z_1(\lambda_1 a) = 0,$$

$$\frac{3-\tau}{1-\tau}(c_{11} - c_{21}) + \frac{1+\tau}{1-\tau}\frac{c_2}{b^2} + \frac{i}{b}[Z_1(\lambda_1 b) - c_{23}R_1(\lambda_2 b)] = 0,$$

$$\frac{3-\tau}{1-\tau}i(c_{11} - c_{21}) - \frac{1+\tau}{1-\tau}i\frac{c_2}{b^2} - \lambda_1 Z_0(\lambda_1 b) + \frac{1}{b}Z_1(\lambda_1 b) + \left[-\frac{R_1(\lambda_2 b)}{b} + \lambda_2 R_0(\lambda_2 b)\right]c_{23} = 0,$$

$$\mu_1\left[\frac{1+\tau}{1-\tau}4i\frac{c_2}{b^3}+\left(\lambda_1^2-\frac{4}{b^2}\right)Z_1(\lambda_1 b)+2\frac{\lambda_1}{b}Z_0(\lambda_1 b)\right]-\mu_2\left[\left(\lambda_2^2-\frac{4}{b^2}\right)R_1(\lambda_2 b)+\frac{2\lambda_2}{b}R_0(\lambda_2 b)\right]c_{23}=0,$$

$$ib\frac{3-\tau}{(1-\tau)^2}\tau^2(\rho_1 c_{11}-\rho_2 c_{21})+\rho_1\left[\frac{i}{b}(1+\tau)\left(\frac{4}{\lambda_1^2 b^2}-\frac{2-4\tau+\tau^2}{(1-\tau)^2}\right)c_2\right.$$
$$\left.+\left(\frac{1-2\tau}{1-\tau}-4\frac{1-\tau}{\lambda_1^2 b^2}\right)Z_1(\lambda_1 b)+2\frac{1-\tau}{\lambda_1 b}Z_0(\lambda_1 b)\right]-\rho_2\left[\left(\frac{1-2\tau}{1-\tau}-4\frac{1-\tau}{\lambda_2^2 b^2}\right)\right.$$
$$\left.\times R_1(\lambda_2 b)+2\frac{1-\tau}{\lambda_2 b}R_0(\lambda_2 b)\right]c_{23}=(\rho_1-\rho_2)\tau^2\frac{\Omega\varepsilon b}{2}, \tag{4.3}$$
$$i(1-\tau)\eta_*=\frac{3-\tau}{1-\tau}\frac{c_{21}}{\Omega}+\frac{i}{\Omega b}R_1(\lambda_2 b)c_{23},\quad Z_n=c_{13}L_n+c_4 M_n,\quad \tau=\omega/\Omega.$$

Using the solution (4.1)-(4.3) of the problem (3.2)-(3.4), we can obtain expressions for the stresses on the inner surface of the cylinder, as well as for the components of the hydrodynamic force with which the liquid filling acts on a unit length of the rotating and precessing cylinder:

$$F_\xi=2\pi a\rho_1\,\mathrm{Re}\,[(1/2)\omega^2 a\varepsilon+2i(\omega+\Omega)c_2/a],$$
$$F_\eta=-4\pi\rho_1(\Omega+\omega)\,\mathrm{Re}\,c_2. \tag{4.4}$$

In Figs. 2 and 3 the solid curves indicate the variation of the force components (4.4) as functions of ω/Ω when $b/a = 0.5$, $\nu_2/\nu_1 = 0.5$, $\rho_2/\rho_1 = 0.5$, $\nu_1/\Omega a^2 = 10^{-5}$ and the scale of the force is $F^* = \pi a^2\rho_1\omega^2\varepsilon$. The character of this variation is due to the resonance excitation of the internal waves in the rotating nonuniform layered liquid filling of the cylinder.

It should be noted that the calculation of the hydrodynamic force within the framework of the nonviscous-liquid model ($\mu_1 = \mu_2 = 0$) yields an identically zero component F_η and an infinitely large component F_ξ for resonance values of ω/Ω. In the case of a viscous liquid, as can be seen from Fig. 2 and Fig. 3, the two components have the same order of magnitude in the vicinity of the resonances.

We can also derive expressions for the components of the hydrodynamic force in the vicinity of the resonance $\omega = \Omega$ obtained from (4.3) and (4.4) by using asymptotic expansions for the cylinder functions at small values of the argument [5]:

$$F_\xi=\pi\rho_1\Omega^2 a^2\varepsilon+O(\Omega-\omega),$$
$$F_\eta=4\pi\varepsilon(\Omega-\omega)\left(1-\frac{\rho_2}{\rho_1}\right)\left(\mu_2\left(1-\frac{b^4}{a^4}\right)+\mu_1\left(1+\frac{b^4}{a^4}\right)\right)\frac{1}{\Delta}+$$
$$+O((\Omega-\omega)^2\ln|\lambda_1 a|), \tag{4.5}$$
$$\Delta=\left(\frac{b^2}{a^2}-1\right)\left(\frac{1}{2}\left(\frac{b^2}{a^2}+1\right)+2\frac{\mu_2}{\mu_1}\left(1-\frac{b^2}{a^2}\right)\left(1-\frac{\nu_1^2}{\Omega^2 a^2 b^2}\right)-\right.$$
$$\left.-\frac{\rho_2}{\rho_1}\left(1+\frac{\mu_2}{\mu_1}\left(1-\frac{b^2}{a^2}\right)\right)\right)+\left(1-\frac{\rho_2}{\rho_1}\right)\left(\left(1+\frac{b^4}{a^4}\right)+\frac{\mu_2}{\mu_1}\left(1-\frac{b^4}{a^4}\right)\right)\ln\frac{a}{b}.$$

When $\rho_2 = 0$, $\mu_2 = 0$, the expressions (4.5) become the corresponding formulas of [3]. From the expressions given it follows that when the resonance $\omega = \Omega$ is passed, the component F_η changes sign. This is in agreement with the concept of "rotational friction" widely known in investigations on the stability of the rotation of rotors with internal friction [6].

5. The Approximate Method for Solving the Hydrodynamic Problem. In the case when the Eckmann numbers $E_k = \nu_k/|\Omega - \omega|a^2$ ($k = 1, 2$) are small and a, b, and (a − b) are of the same order of magnitude, the solution of the hydrodynamic problem formulated in Sec. 3 can be represented as the sum of two components:

$$u_k=u_k^{(1)}+u_k^{(2)},\quad v_k=v_k^{(1)}+v_k^{(2)},\quad p_k=p_k^{(1)}+p_k^{(2)}. \tag{5.1}$$

The large-scale first component $u_k^{(1)}$, $v_k^{(1)}$, $p_k^{(1)}$ has a spatial scale of the order of a and approximately describes the motion of the liquid in the main volume; the small-scale second component $u_k^{(2)}$, $v_k^{(2)}$, $p_k^{(2)}$ forms boundary layers on the inner surface of the cylinder and on the interface between the liquids. The characteristic spatial scale of the second component, as can be seen from dimensional considerations, is of the order of $(\nu_k/|\Omega - \omega|)^{1/2}$.

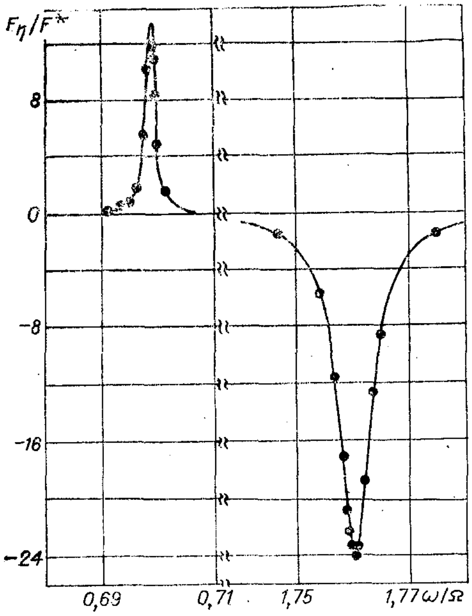

Fig. 3

The large-scale component of the solution satisfies the system of equations obtained from (3.2) if we set $\nu_k = 0$:

$$(\Omega-\omega)\frac{\partial u_k^{(1)}}{\partial\varphi} = \Omega^2 r + \omega^2\varepsilon\cos\varphi + 2\Omega v_k^{(1)} - \frac{1}{\rho_k}\frac{\partial p_k^{(1)}}{\partial r},$$
$$(\Omega-\omega)\frac{\partial v_k^{(1)}}{\partial\varphi} = -2\Omega u_k^{(1)} - \omega^2\varepsilon\sin\varphi - \frac{1}{\rho_k r}\frac{\partial p_k^{(1)}}{\partial\varphi}, \quad \frac{\partial}{\partial r} r u_k^{(1)} + \frac{\partial}{\partial\varphi} v_k^{(1)} = 0. \tag{5.2}$$

We obtain approximate equations for the small-scale component of the solution in boundary layers on the inner surface of the cylinder and on the interface between the liquids. We substitute into (3.2) a solution in the form (5.1), where $u_k^{(1)}$, $v_k^{(1)}$, $p_k^{(1)}$ satisfy (5.2) and $u_k^{(2)} = E_k^{1/2}\bar{u}_k^{(2)}$, $v_k^{(2)} = \bar{v}_k^{(2)}$, $p_k^{(2)} = \bar{p}_k^{(2)}$. We make the substitution $r_0 - r = E_k^{1/2}\bar{r}$, $\varphi = \bar{\varphi}$, $r_0 = a$, b, and letting E_k approach zero, we leave the leading terms in the equations (3.2), assuming that $\bar{u}_k^{(2)}$, $\bar{v}_k^{(2)}$, $\bar{p}_k^{(2)}$ and their derivatives with respect to $\bar{r}$, $\bar{\varphi}$ are of zero order with respect to E_k. As a result, we have

$$\frac{\partial p_k^2}{\partial r} = 0, \quad (\Omega-\omega)\frac{\partial v_k^{(2)}}{\partial\varphi} = -\frac{1}{r_0\rho_k}\frac{\partial p_k^{(2)}}{\partial\varphi} + \nu_k\frac{\partial^2 v_k^{(2)}}{\partial r^2},$$
$$\frac{\partial u_k^{(2)}}{\partial r} + \frac{1}{r_0}\frac{\partial v_k^{(2)}}{\partial\varphi} = 0. \tag{5.3}$$

Application of the standard procedure of boundary-layer theory presupposes first of all finding the large-scale component of the solution starting from the equations (5.2) with the boundary condition that the inner surface of the cylinder is impermeable:

$$u_1^{(1)} = 0, \quad r = a, \tag{5.4}$$

and with the conditions of continuity of the normal components of velocity and pressure and the kinematic condition on the interface between the liquids:

$$u_1^{(1)} = u_2^{(1)}; \tag{5.5}$$
$$p_2^{(1)} - p_1^{(1)} + \Omega^2 b\eta(\rho_2-\rho_1) = 0; \tag{5.6}$$
$$(\Omega-\omega)\frac{\partial\eta}{\partial\varphi} = u_1^{(1)} \quad \text{for} \quad r = b, \tag{5.7}$$

and then solving the actual equations of the boundary layer (or finding the small-scale component of the solution). The previously found large-scale component of the solution of the problem is used in formulating the boundary conditions for the boundary-layer equations. Such a procedure, however, does not enable us to find the solution in a neighborhood of the wave resonances, since for resonance values of ω/Ω the solution of the nonviscous problem for the large-scale component becomes infinitely large. The nature of this difficulty lies in the fact that in a neighborhood of the wave resonances we must not disregard the reaction of the boundary layer on the large-scale component of the solution. In what follows, we shall propose a modification of the standard procedure of boundary-layer theory which makes it possible to get around the difficulty.

We shall assume as an approximation that the boundary layers on the inner surface of the cylinder and at the interface between the liquids are of finite thickness, while outside the boundary layers the small-scale component is negligibly small.

From the third equation of the system (5.3) with $r_0 = a$, we have

$$u_1^{(2)} = -\frac{1}{a}\frac{\partial}{\partial\varphi}\int_{a-\delta}^{r} v_1^{(2)}dr, \quad a-\delta\leqslant r\leqslant a, \tag{5.8}$$

where δ is the thickness of the boundary layer at the surface of the cylinder. Substituting (5.8) into the boundary condition for the normal component of the velocity at the surface of the cylinder $u_1^{(1)} + u_1^{(2)} = 0$, $r = a$ and introducing the thickness δ_a^*, defined by the expression

$$\int_{a-\delta}^{a} v_1^{(2)}dr = -\delta_a^* v_1^{(1)}(a, \varphi-\varphi^*), \quad \varphi^* = \frac{\pi}{4}\frac{\Omega-\omega}{|\Omega-\omega|}, \tag{5.9}$$

we obtain

$$u_1^{(1)}(a, \varphi) = -\frac{1}{a}\frac{\partial}{\partial\varphi}\delta_a^* v_1^{(1)}(a, \varphi-\varphi^*). \tag{5.10}$$

We take (5.10) as the boundary condition for the large-scale component of the solution for $r = a$ instead of the condition (5.4).

Similarly, from the third equation of the system (5.3) when $r_0 = b$, we obtain

$$\begin{gathered} u_1^{(2)} = \frac{1}{b}\frac{\partial}{\partial\varphi}\int_{r}^{b+\delta_1} v_1^{(2)}dr, \quad b\leqslant r\leqslant b+\delta_1, \\ u_2^{(2)} = -\frac{1}{b}\frac{\partial}{\partial\varphi}\int_{b-\delta_2}^{r} v_2^{(2)}dr, \quad b-\delta_2\leqslant r\leqslant b, \end{gathered} \tag{5.11}$$

where δ_1, δ_2 are the thickness of the boundary layers at the interface between the liquids. Substituting (5.11) into the condition of continuity of the normal component of the velocity at the interface between the liquids $u_1^{(1)} + u_1^{(2)} = u_2^{(1)} + u_2^{(2)}$, $r = b$ and introducing the thickness δ_b^*, defined by the expression

$$\int_{b}^{b+\delta_1} v_1^{(2)}dr + \int_{b-\delta_2}^{b} v_2^{(2)}dr = \delta_b^*\left(v_1^{(1)}(b, \varphi-\varphi^*) - v_2^{(1)}(b, \varphi-\varphi^*)\right), \tag{5.12}$$

we obtain

$$u_1^{(1)}(b, \varphi) - u_2^{(1)}(b, \varphi) = -\frac{1}{b}\frac{\partial}{\partial\varphi}\delta_b^*\left(v_1^{(1)}(b, \varphi-\varphi^*) - v_2^{(1)}(b, \varphi-\varphi^*)\right). \tag{5.13}$$

We take (5.13) as the boundary condition for the large-scale component of the solution when $r = b$ instead of (5.5).

The kinematic condition at the interface between the liquids, taking account of (5.1) and (5.11), can be written as

$$(\Omega-\omega)\frac{\partial\eta}{\partial\varphi} = u_1^{(1)} + \frac{1}{b}\frac{\partial}{\partial\varphi}\int_{b}^{b+\delta_1} v_1^{(2)}dr, \quad r = b.$$

Introducing the thickness δ_1^*, defined by the expression

$$\int_b^{b+\delta_1} v_1^{(2)} dr = -\delta_1^* \left(v_1^{(1)}(b, \varphi - \varphi^*) - v_2^{(1)}(b, \varphi - \varphi^*)\right), \tag{5.14}$$

we obtain

$$(\Omega - \omega)\frac{\partial \eta(\varphi)}{\partial \varphi} = u_1^{(1)}(b, \varphi) - \frac{1}{b}\frac{\partial}{\partial \varphi}\delta_1^* \left(v_1^{(1)}(b, \varphi - \varphi^*) - v_2^{(1)}(b, \varphi - \varphi^*)\right). \tag{5.15}$$

We take (5.15) as the generalization of the kinematic condition (5.7) of the problem for the large-scale component of the solution. The condition (5.6) for the large-scale component when $r = b$ is retained without any change.

The thickness δ_a^*, δ_b^*, δ_1^* appearing in (5.10), (5.13), and (5.15) can be found by using the solution of the boundary-value problem for the small-scale component with the equations (5.3), the boundary conditions

$$\begin{gathered} v_1^{(2)} = -v_1^{(1)}, \quad r = a, \quad v_1^{(2)} - v_2^{(2)} = v_2^{(1)} - v_1^{(1)}, \\ \mu_1 \frac{\partial}{\partial r} v_1^{(2)} = \mu_2 \frac{\partial}{\partial r} v_2^{(2)}, \quad r = b, \end{gathered} \tag{5.16}$$

and the condition that the small-scale component is negligibly small outside the boundary layers. The third condition of (5.16) is obtained as $E_k \to 0$ from the condition of continuity of the tangential stress at the interface between the liquids when we make the additional assumption that near the interface $v_k^{(1)}$ and $v_k^{(2)}$ are of the same order of magnitude. The condition that the small-scale component of the solution is negligibly small outside the boundary layers, together with the first equation of the system (5.3), yields $p_k^{(2)} \equiv 0$. Consequently the boundary-value problem for the small-scale component reduces to integrating the equations

$$(\Omega - \omega)\frac{\partial v_k^{(2)}}{\partial \varphi} = \nu_k \frac{\partial^2 v_k^{(2)}}{\partial r^2} \quad (k = 1, 2) \tag{5.17}$$

with the boundary conditions (5.16), the condition that $v_k^{(2)}$ is negligibly small outside the boundary layers, and then calculating $u_k^{(2)}$ by the formulas (5.8) and (5.11).

Thus, the proposed modification of the standard procedure of boundary-layer theory consists in the fact that the boundary-value problem for the large-scale component of the solution is posed with the equations (5.2) and the boundary conditions (5.6), (5.10), (5.13), (5.15). This problem for $\delta_a^* = \delta_b^* = \delta_1^* = 0$ becomes the nonviscous problem (5.2), (5.4)-(5.7), which usually precedes the consideration of the motion of a liquid in boundary layers.

Let us now consider the actual construction of the approximate solution of the boundary value problem formulated in Sec. 3 in the case when the Eckmann numbers E_1 and E_2 are small and a, b, and $(a - b)$ are of the same order of magnitude. We shall try to find it in the form (5.1), setting

$$\begin{gathered} u_k^{(1),(2)} = 2\,\mathrm{Re}\, u_k^{*(1),(2)}(r)\, e^{i\varphi}, \quad v_k^{(1),(2)} = 2\,\mathrm{Re}\, v_k^{*(1),(2)}(r)\, e^{i\varphi}, \\ \frac{p_k^{(1)}}{\rho_k} = 2\,\mathrm{Re}\, p_k^{*(1)}(r)\, e^{i\varphi} + \frac{\Omega^2 r^2}{2} + \text{const}, \quad k = 1, 2. \end{gathered}$$

From the equations (5.17) with the boundary conditions (5.16) we obtain, by using the condition that the small-scale component of the solution is negligibly small outside the boundary layers, the following:

$$\begin{gathered} v_1^{(2)} = -2\,\mathrm{Re}\left(v_1^{*(1)}(a)\, e^{i\lambda_1(a-r)+i\varphi}\right), \quad a - \delta < r \leqslant a, \\ v_1^{(2)} = -\frac{2}{1+d}\mathrm{Re}\left[\left(v_1^{*(1)}(b) - v_2^{*(1)}(b)\right) e^{i\lambda_1(r-b)+i\varphi}\right], \quad b \leqslant r \leqslant b + \delta_1, \\ v_2^{(2)} = \frac{2d}{1+d}\mathrm{Re}\left[\left(v_1^{*(1)}(b) - v_2^{*(1)}(b)\right) e^{i\lambda_2(b-r)+i\varphi}\right], \quad b - \delta_2 \leqslant r \leqslant b, \\ d = (\mu_1\rho_1/\mu_2\rho_2)^{1/2}, \quad k = 1, 2. \end{gathered} \tag{5.18}$$

The expressions for $u_1^{(2)}$, $u_2^{(2)}$ are obtained from (5.8) and (5.11), making use of (5.18). The condition for negligibility of the small-scale component of the solution outside the boundary layers will be satisfied if the provisional thicknesses of the boundary layers are chosen so as to satisfy the inequalities

$$\delta_1,\ \delta >> (2\nu_1/|\Omega-\omega|)^{1/2},\quad \delta_2 \gg (2\nu_2/|\Omega-\omega|)^{1/2}.$$

Having obtained the expressions (5.18), we find from (5.9), (5.12), and (5.14) that

$$\delta_a^* = (\nu_1/|\Omega-\omega|)^{1/2},\quad \delta_b^* = \frac{1}{1+d}\left(\frac{\rho_1}{\rho_2}-1\right)\delta_a^*,\quad \delta_1^* = \delta_a^*/(1+d).$$

Integrating (5.2) with the boundary conditions (5.6), (5.10), (5.13), (5.15), we obtain

$$u_k^{*(1)} = A_k + B_k/r^2,\quad v_k^{*(1)} = i\,(A_k - B_k/r^2),$$
$$p_k^{*(1)} = i\,[(\Omega+\omega)A_k + (3\Omega-\omega)B_k/r^2]\,r,\quad k = 1, 2,$$

where the constants A_k and B_k satisfy the system of algebraic equations

$$(1-g_a)A_1 + (1+g_a)\frac{B_1}{a^2} = 0,\quad g_a = \frac{\delta_a^*}{a}e^{-i\varphi*},$$

$$(1-g_b)A_1 + (1+g_b)\frac{B_1}{b^2} - (1-g_b)A_2 = 0,\quad g_b = \frac{\delta_b^*}{b}e^{-i\varphi*},$$

$$[-i(\Omega+\omega) + q(1+g_1)]A_1 + [-i(3\Omega-\omega) + q(1-g_1)]\frac{B_1}{b^2} + \left[i(\Omega+\omega)\frac{\rho_2}{\rho_1} - qg_1\right]A_2 = \left(1-\frac{\rho_2}{\rho_1}\right)\frac{\omega^2\varepsilon}{2}, \tag{5.19}$$

$$q = i\frac{\Omega^2}{\Omega-\omega}\left(1-\frac{\rho_2}{\rho_1}\right),\quad g_1 = \frac{\delta_1^*}{b}e^{-i\varphi*},\quad B_2 = 0.$$

Using the approximate expressions for the hydrodynamic elements, we can calculate the stresses on the surface of the cylinder and the force with which the liquid filling acts on a unit length of the rotating and processing cylinder:

$$F_\xi = 2\pi a^2\rho_1 \operatorname{Re}(\omega^2\varepsilon/2 + iD),\quad F_\eta = -2\pi a^2\rho_1 \operatorname{Re} D,$$
$$D = 2[\Omega A_1 + (2\Omega-\omega)B_1/a^2]. \tag{5.20}$$

In Figs. 2 and 3 the dots show how the hydrodynamic force components F_ξ/F^*, F_η/F^* vary with ω/Ω, as determined in accordance with (5.19) and (5.20) for the parameter values indicated in the example given in Sec. 4. It can be seen that the approximate results are in good agreement with the exact values.

The proposed modification of the standard procedure of boundary-layer theory simplifies the calculation of the hydrodynamic force components F_ξ, F_η for small E_1, E_2 and is especially effective in the case when the cylinder is filled with more than two immiscible liquids.

It should be noted that when $\omega = \Omega$, the thicknesses of the boundary layers become infinitely large and the approximate constructions described above are not applicable. However, a comparison with the exact solution shows that in a neighborhood of the resonance $\omega = \Omega$ the variation of the force components F_ξ, F_η as functions of ω/Ω can be found with sufficient accuracy by interpolation between the extreme values, taking account of the fact that when $\omega/\Omega = 1$, F_ξ has a minimum and F_η passes through zero.

6. Boundaries of the Regions with Different Degrees of Instability in the Space of Problem Parameters. This, as has already been noted in Sec. 2, can be found from the conditions of realizability of circular precession of a rotating cylinder with a liquid filling. These conditions can be obtained by substituting into the right sides of (1.1) $F_1 = F_\xi\cos\omega t - F_\eta\sin\omega t$, $F_2 = F_\xi\sin\omega t + F_\eta\cos\omega t$ and substituting into the left sides of those equations the expressions $x_1^0 = \varepsilon\cos\omega t$, $x_2^0 = \varepsilon\sin\omega t$. As a result, we shall have

$$\frac{F_\xi}{F^*}\left(\frac{\omega}{\Omega}\right)^2 = K_* - \frac{M}{m}\left(\frac{\omega}{\Omega}\right)^2,\quad \frac{F_\eta}{F^*}\left(\frac{\omega}{\Omega}\right)^2 = H_*\frac{\omega}{\Omega},$$
$$K_* = K/m\Omega^2,\quad H_* = H/m\Omega,\quad m = \pi a^2\rho_1. \tag{6.1}$$

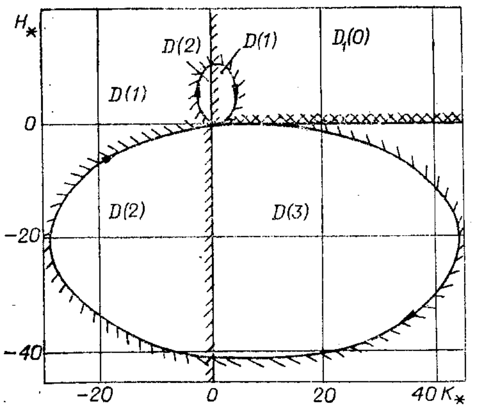

Fig. 4

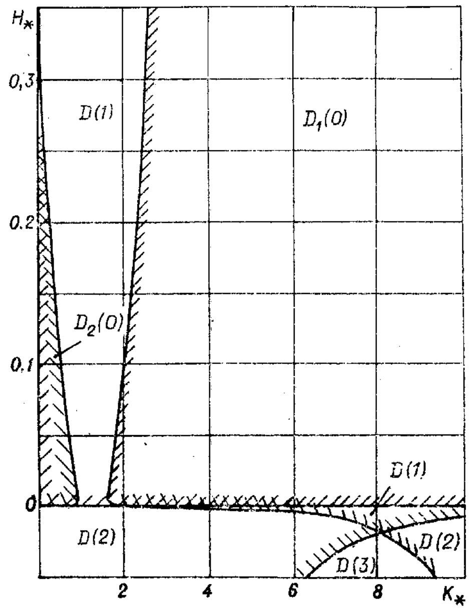

Fig. 5

For fixed $\nu_1/\Omega a^2$, $(a - b)/a$, ρ_2/ρ_1, μ_2/μ_1, M/m, the conditions (6.1) parametrically define a curve (with parameter ω/Ω along the curve) which divides the plane of the parameters of gripping of the cylinder axis H_*, K_* into regions D(n) with degree of instability n. This curve (the D curve) is completely analogous to the one constructed in [3]. In Figs. 4 and 5 we show the subdividing process carried out by the D curve when $b/a = 0.5$, $\nu_1/\Omega a^2 = 10^{-5}$, $\mu_2/\mu_1 = 0.25$, $\rho_2/\rho_1 = 0.5$, M/m = 1.68. The regions of stability are denoted by $D_1(0)$ and $D_2(0)$.

APPENDIX

In the case of circular precession with small radius the point of intersection of the cylinder axis with the plane $O_1x_1x_2$ describes a circle with some frequency ω, and the hydrodynamic fields vary with time at the same frequency in the system $O_1x_1x_2z$. The system $O\xi\eta$ undergoes periodic motion at frequency ω with respect to Ox_1x_2. Consequently in the case of circular precession with small radius the motion of the liquids in the system $O\xi\eta$ must satisfy the condition for periodicity in time with frequency ω. This condition is satisfied by the time-independent motion of the liquids in $O\xi\eta$ (let us call it A), which was considered in Sec. 4. Suppose that together with this in the case of circular precession with small radius we can also have a second motion of the liquids in the system $O\xi\eta$ (let us call it A') which is periodically dependent on time with frequency ω. The differences between the hydrodynamic fields of the motions A and A' satisfy a system of homogeneous Navier–Stokes equations in the plane $O\xi\eta$ linearized near quasisolid rotation of the liquids (3.1), from which, using the boundary conditions, we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\sum_{k=1}^{2}\int_{\tau_h}\rho_k \mathbf{w}^2 d\tau_h + (\rho_1-\rho_2)\frac{\Omega^2 b^2}{2}\int_0^{2\pi}\zeta^2 d\varphi\right) = -\sum_{k=1}^{2}\mu_k\int_{\tau_k}\left[2\left(\frac{\partial w_\xi}{\partial\xi}\right)^2 + 2\left(\frac{\partial w_\eta}{\partial\eta}\right)^2 + \left(\frac{\partial w_\xi}{\partial\eta} + \frac{\partial w_\eta}{\partial\xi}\right)^2\right]d\tau_k, \quad (A.1)$$

where $\mathbf{w} = \mathbf{e}_\xi w_\xi + \mathbf{e}_\eta w_\eta$ is the difference between the velocity fields of the motions A and A'; $\zeta(\varphi, t)$ is the distance along the radius drawn from the point O between the interfaces of the liquids in the motions A and A'; τ_1 is the annulus $a \geqslant r \geqslant b$, τ_2 is the circular disk $0 \leqslant r \leqslant b$ in the plane $O\xi\eta$. From (A.1) it can be seen that the differences between the hydrodynamic fields of the motions A and A' cannot be periodically dependent on time, and this contradicts the condition of periodicity of the motions A and A' themselves. We are left to conclude that in the case of circular precession of the rotating cylinder with a liquid filling along a circle of small radius, the motion of the filling liquids with respect to the system $O\xi\eta$ is independent of time.

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FLOW SYMMETRY DISTURBANCE DUE TO THERMAL INSTABILITY

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The instability of the laminar mode associated with the origin of turbulence is usually inessential for flows of a strongly viscous fluid since the Reynolds numbers are small. In the forefront for such flows is the thermal instability detected and investigated in [1, 2] and elsewhere. It was shown in [2] that the thermal instability holds for the pressure drop Δp determined and results in jumps in the flow rate for definite critical values of Δp (hydrodynamic inflammation and extinction).

Meanwhile, a thermal instability also occurs in electrical systems even for a fixed current (the analog of the mass flow in hydrodynamics), where the development of the instability results in inhomogeneous [3] or nonsymmetric [4] modes.

Disturbance of the flow symmetry through a pair of tubes (connected in parallel) is considered in this paper for a fixed total mass flow rate. It is shown that in contrast to an analogous electrical system [4], the disturbed symmetry is restored in the case under consideration as the flow rate increases. Restoration of the symmetry is due to the convective nature of the instability.

1. The flow of a strongly viscous incompressible fluid through two cylindrical tubes connected in parallel and with a given total flow rate is considered. As the fluid moves, heat is liberated because of dissipation and is eliminated in the tube walls.

The following equations hold for tubes connected in parallel

$$\frac{\pi r^4}{8}\Delta p = Q_1\int_0^l \mu(T_1)\,dz = Q_2\int_0^l \mu(T_2)\,dz, \quad Q = Q_1 + Q_2,$$

where Δp is the pressure drop between the entrance into, and exit from the tubes, r is the tube radius, l is the tube length, z is the coordinate along the tube axes, μ is the dynamic viscosity, T_1 and T_2 are the fluid temperatures, Q_1 and Q_2 are the mass flow rates, the subscripts 1 and 2 refer, respectively, to the first and second tubes, and Q is the total mass flow rate which is a given quantity. As regards the equation in the temperature, then under the conditions

$$\mathrm{Pr} = \mu/(\rho\chi) \gg 1,\ \mathrm{Pe} = Ql/(\pi r^2\chi) \gg 1,\ \mathrm{Bi} = \alpha r/(c\rho\chi) \ll 1,$$

where χ is the thermal diffusivity coefficient, ρ is the density, c is the specific heat of the fluid, α is the coefficient of heat transfer, Pr is the Prandtl criterion, Pe is the Peclet criterion, Bi is the Biot criterion, they take a form analogous to (1.11) in [2] for both tubes. In dimensionless variables

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